Approximation in Multiobjective Optimization

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Abstract. Some results of approximation type for multiobjective optimization problems with a finite number of objective functions are presented. Namely, for a sequence of multiobjective optimization problems P_n which converges in a suitable sense to a limit problem P, properties of the sequence of approximate Pareto efficient sets of the P_n's, are studied with respect to the Pareto efficient set of P. The exterior penalty method as well as the variational approximation method appear to be particular cases of this framework.

Key words. Epiconvergence, ϵ-efficient set, Mosco-convergence, multiobjective optimization, Pareto, penalization, variational approximation.

AMS subject classifications (1991). 65K05, 65K10, 90C31.

Introduction

This paper deals with the approximation of the efficient solutions in the (strong or weak) sense of Pareto, of a multiobjective optimization problem (objective taking values in R^p) by approximate efficient solutions of approximate multiobjective optimization problems.

The notion of ϵ-efficiency ($\epsilon \in R^p$) for a given multiobjective optimization problem is analogous to the notion of ϵ-suboptimality (ϵ real) in scalar optimization ($p = 1$). Some properties are presented in Section 1.

For approximating a multiobjective optimization problem by "more simple" ones, we introduce in Section 2 a convergence notion which preserves efficiency and reduces to epiconvergence in the scalar case. Variational properties of epiconvergence are thus extended to multiobjective optimization. Convergence of the exterior penalty method and of the variational approximation method in the multiobjective case can be obtained as applications of the main results of Section 2. This is done in Section 3 and 4 respectively.

1. Approximate Efficiency

Let $Y = R^p$ ordered by the usual product order cone $Y_+ = R^p_+$. For the various ordering relationships between two elements of Y we shall use the following notations

$$y^1 \geqq y^2 \quad \text{iff } y^1 - y^2 \in Y_+$$
$$y^1 \geqslant y^2 \quad \text{iff } y^1 - y^2 \in Y_+ / \{0\}$$
$$y^1 > y^2 \quad \text{iff } y^1 - y^2 \in int(Y_+)$$

Opposite relations are noted like $\not\geqslant$ and $\not>$.

Moreover, we denote $y^1.y^2 = \Sigma_{i=1}^p y_i^1 y_i^2$ the euclidian inner product on Y and $|.|$ the associated norm.

A multiobjective optimization problem (MOP) is defined by a nonempty set C (the feasible set) and a function $f: C \to Y$ (the (multi)objective). First, let us recall the definition of approximate efficiency (also called epsilon efficiency) introduced in 1979 by Kutateladze [7] and used in [12] for computational purposes and in [10] in the context of duality theory in vector optimization. This definition makes precise what is understood by approximately minimizing the objective.

DEFINITION 1.1. Let $P = (C, f)$ be a MOP and $\epsilon \in Y$. Define

$$\epsilon - E_s(P) := \{\bar{x} \in C; \forall x \in C,\ f(\bar{x}) \not\geqslant f(x) + \epsilon\}\ ,$$

$$\epsilon - E_w(P) := \{\bar{x} \in C; \forall x \in C,\ f(\bar{x}) \not> f(x) + \epsilon\}\ .$$

REMARK 1.1.

1. If $\epsilon = 0$, $E_s(P) := 0 - E_s(P)$ (resp. $E_w(P) := 0 - E_w(P)$) is the strong (resp. weak) Pareto efficient set of P.
2. Although, as in the scalar case, local efficient solutions can be considered, this paper deals only with global efficient solutions and thus relates to global optimization.
3. Denoting by $f_i: C \to R$ (resp. ϵ_i), $i = 1, \dots, p$, the components of f (resp. ϵ), $\bar{x} \in C$ is an ϵ-efficient solution in the strong sense of Pareto of problem P (or more simply, $\bar{x}$ is ϵ-strong Pareto), i.e., $\bar{x} \in \epsilon - E_s(P)$, if and only if there exists no x in C which improves each f_i no less than ϵ_i and some f_i more than ϵ_i, or equivalently, for every x in C, if x improves some f_i more than ϵ_i, there exists $j \neq i$ such that the improvement of f_j is lesser than ϵ_j.
   In the same way, $\bar{x} \in C$ is an ϵ-efficient solution in the weak sense of pareto of problem P (or more simply, $\bar{x}$ is ϵ-weak Pareto), i.e., $\bar{x} \in \epsilon - E_w(P)$, if and only if there exists no x in C which improves each f_i more than ϵ_i, or equivalently, for every x in C, there exists some i such that the improvement of f_i is not greater than ϵ_i.
4. If $p = 1$,

$$\begin{aligned}\epsilon - E_s(P) &= \epsilon - E_w(P)\\ &= \epsilon - Argmin_C f := \{\bar{x} \in C;\ f(\bar{x}) \leqslant inf_C f + \epsilon\}\ .\end{aligned}$$

   Thus ϵ-efficiency reduces to ϵ-suboptimality in the scalar case.

The following simple results clarify the relationships between the different notions introduced above. Omitted proofs are trivial.

PROPOSITION 1.1. *We have (see figure for an illustration)*

1. $\forall \epsilon \in Y$

$$\epsilon - E_s(P) = f^{-1}\{f(C) \cap [Y/(f(C) + Y_+/\{0\} + \epsilon)]\},$$
$$\epsilon - E_w(P) = f^{-1}\{f(C) \cap [Y/(f(C) + int(Y_+) + \epsilon)]\}.$$

2. $\epsilon^1 \geqq \epsilon^2 \Rightarrow \epsilon^2 - E_s(P) \subset \epsilon^1 - E_s(P)$ *and* $\epsilon^2 - E_w(P) \subset \epsilon^1 - E_w(P)$.
3. $\forall \epsilon \in Y, \epsilon - E_s(P) \subset \epsilon - E_w(P)$.
4. $\epsilon^2 > \epsilon^1 \Rightarrow \epsilon^1 - E_w(P) \subset \epsilon^2 - E_s(P)$.
   In particular, $\forall \epsilon > 0$, $E_w(P) \subset \epsilon - E_s(P)$.
5. $$E_s(P) = \bigcap_{\epsilon \geqslant 0} \epsilon - E_s(P), \quad E_w(P) = \bigcap_{\epsilon > 0} \epsilon - E_w(P).$$

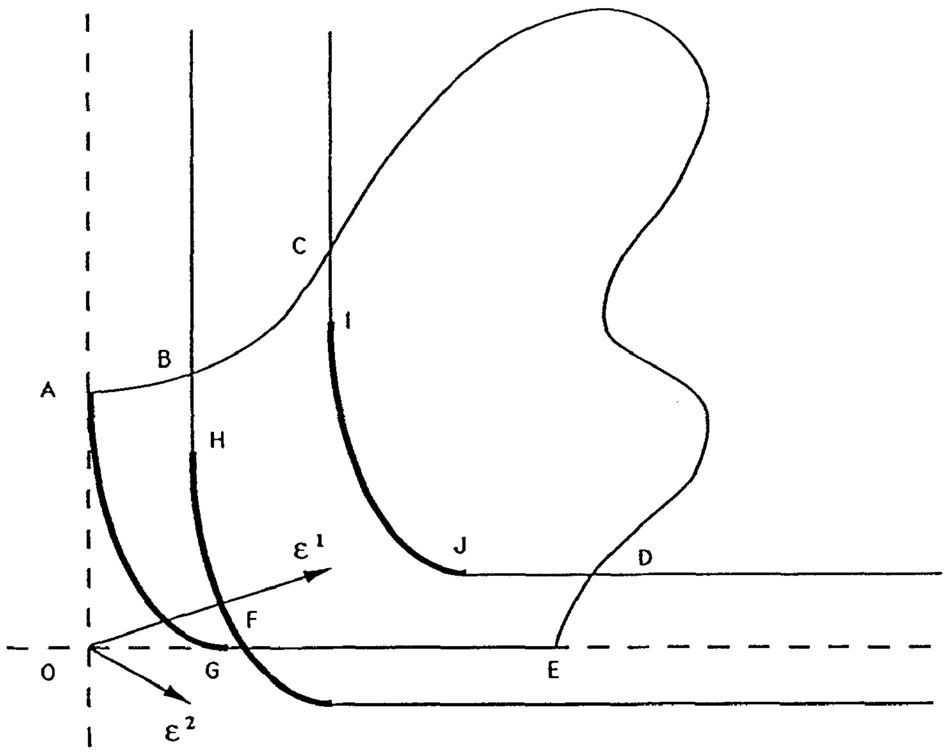

Fig. 1.
$X = Y = R^2$, f = identity
C = closed area limited by the curved line $ABCDEFGA$
$E_s(P)$ = closed line AG, $E_w(P$ = closed line AGE,
$\epsilon^1 - E_w(P)$ = closed area limited by $ABCIJDEFGA$.
$\epsilon^1 - E_s(P) = \epsilon^1 - E_w(P)$ expected segments $[C, I[$ and $[D, J[$.
$\epsilon^2 - E_w(P)$ = closed area limited by $ABHFGA$.
$\epsilon^2 - E_s(P) = \epsilon^2 - E_w(P)$ excepted segments $[B, H[$ and $]G, F]$.

Proof. We just give the proof of 5 for the strong case. Let $\bar{x} \in C$. We must prove

$$\bar{x} \notin E_s(P) \Leftrightarrow \exists \epsilon \geqslant 0\,, \quad \bar{x} \notin \epsilon - E_s(P)$$

or

$$\exists x \in C\,, \quad f(\bar{x}) - f(x) \geqslant 0 \Leftrightarrow \exists \epsilon \geqslant 0\,, \quad \exists x \in C\,, \quad f(\bar{x}) - f(x) \geqslant \epsilon$$

which is truc (for the implication to the right take $\epsilon = (f(\bar{x}) - f(x))/2$). □

PROPOSITION 1.2. 1. *Let $\epsilon \in Y$ and $y \in Y$. Then*

$$y > (resp. \geqslant) 0 \Leftrightarrow y.\epsilon - Argmin\ P_y \subset \epsilon - E_s(P)\ (resp.\ \epsilon - E_w(P))$$

where

$$y.\epsilon - Argmin\ P_y := \{\bar{x} \in C;\ y.f(\bar{x}) \leqslant y.f(x) + y.\epsilon, \forall x \in C\}\,.$$

2. *Let $\alpha \in R$, $y \in Y/\{0\}$ and $\epsilon = \alpha y / |y|^2$. Then*

$$y > (resp. \geqslant) 0 \Rightarrow \alpha - Argmin\ P_y \subset \epsilon - E_s(P)\ (resp.\ \epsilon - E_w(P))\,.$$

3. *If C is a convex subset of a vector space and if each f_i is convex, then for every $\bar{x} \in \epsilon - E_w(P)$ there exists $y \geqslant 0$ such that $\bar{x} \in y.\epsilon - Argmin\ P_y$.*

Proof. 1. Immediate by contradiction. 2. Direct consequence of 1. 3. The same as $\epsilon = 0$ ([1], proposition 4, p. 298, or [6]). □

The following proposition deals with the question of existence of inexact efficient points.

PROPOSITION 1.3. *We have*

1. $\forall \epsilon \leqslant (resp. < 0,\ \epsilon - E_s(P)\ (resp.\ \epsilon - E_w(P)) = \emptyset$.
2. *Let $\epsilon \neq 0$. If f is bounded from below then*

$$\epsilon \not\leqslant (resp. \not<) 0 \Rightarrow \epsilon - E_s(P)\ (resp.\ \epsilon - E_w(P)) \neq \emptyset\,.$$

Proof. We just give the proof for the strong case.

1. If $E_s(P) \neq \emptyset$, then

$$\forall \bar{x} \in E_s(P),\ f(\bar{x}) \not\geqslant f(\bar{x}) + \epsilon, \text{ i.e., } 0 \not\geqslant \epsilon\,.$$

2. $\forall y \geqq 0$ the real function $y.f(.)$ is bounded from below on C. Furthermore, we can find some $y > 0$ such that $y.\epsilon > 0$. Therefore $y.\epsilon - Argmin(P_y) \neq \emptyset$. Then apply Proposition 1.2. □

REMARK 1.2. If $\bar{x} \in \epsilon - E_s(P)$ (resp. $\epsilon - E_w(P)$) for some $\epsilon \in Y$, by Proposition 1.1 (2.), there exists $\epsilon' \geqq 0$ such that $\bar{x} \in \epsilon' - E_s(P)$ (resp. $\epsilon' - E_w(P)$).

PROPOSITION 1.4. *Let $P = (C, f)$ be a MOP in which C is assumed to be* finite. *Then*

$$\exists \epsilon > 0 \text{ such that } \epsilon - E_w(P) = E_w(P) .$$

Proof. Let $\Gamma = \{(x, x') \in C \times C; \quad f(x) > f(x')\}$.
If $\Gamma = \emptyset$, then

$$\forall \epsilon \geqq 0 , \quad E_w(P) = \epsilon - E_w(P) = C .$$

If $\Gamma \neq \emptyset$, let

$$\gamma = \min_{(x,x') \in \Gamma} \min_i (f_i(x) - f_i(x')) .$$

Notice that $\gamma > 0$. Let $\bar{x} \in C$, $\bar{x} \not\in E_w(P)$. Then

$$\exists x \in C , \quad f(\bar{x}) > f(x)$$

which implies

$$f(\bar{x}) > f(x) + (f(\bar{x}) - f(x))/2 .$$

Therefore, $\bar{x} \not\in \epsilon - E_w(P)$ with $\epsilon_i := \gamma/2$, $i = 1, \ldots, p$. □

REMARK 1.3. The analogous result for strong Pareto efficiency is not true in general as it is proved by the following counter-example

$$Y = R^2 , \quad C = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset R^2 , \quad f(x) = x .$$

Then $E_s(P) = \{(0, 0)\}$ and for all $\epsilon \not\leq 0$, $\epsilon - E_s(P)$ contains at least one of the three other points of C.

2. Main Approximation Results

Hereafter, the feasible sets of all considered MOPS are assumed to be subsets of the same topological space X. For $D \subset X$ and $g: D \to R$, we shall denote by $\bar{g}$ the extension of g by $+\infty$ outside D.

DEFINITION 2.1. A sequence of MOPS (C_n, f^n) converges to the MOP (C, f) and we note $(C_n, f^n) \to (C, f)$ iff the two following sentences hold

$$\forall x \in C , \quad \exists \{x_n\} \subset X, \text{ such that } \forall n \; x_n \in C_n , \; \lim_{n\to\infty} x_n = x \text{ and}$$
$$\forall i \; \limsup_{n\to\infty} f^n_i(x_n) \leq f_i(x) , \tag{1}$$

$$\forall x \in X \quad \forall \{x_n\} \subset X \text{ such that } \lim_{n\to\infty} x_n = x, \text{ then}$$
$$\forall i \; \liminf_{n\to\infty} \overline{f^n_i}(x_n) \geq \bar{f}_i(x) , \tag{2}$$

recalling that when $x \not\in C$ (resp. $x_n \not\in C_n$)$\bar{f}_i(x)$ (resp. $\bar{f}^n_i(x_n)$) $= +\infty$.

REMARK 2.1.

1. If $p=1$, this notion of convergence reduces to the one of (sequential) epiconvergence which during the past twenty-five years, has had many applications in the limit analysis of sequences of variational problems [2] and in perturbation of iterative methods for scalar optimization [8].

2. If $p \geq 2$, this notion is stronger than the epiconvergence of each sequence of scalar optimization problems $\{(C_n, f_i^n)\}$. Namely, in the first sentence (1) the sequence $\{x_n\}$ must be common to each sequence of scalar problem. Nevertheless

3. If (C_n, g_0^n) is a sequence of scalar $(g_0^n: C_n \rightarrow R)$ optimization problems which epiconverges to the scalar optimization problem (C, g_0) we get trivially that the sequence of MOPS $\{(C_n, g_i^n := g_0^n, i=1, \ldots, p)\}$ converges in the above sense to the MOP $(C, g_i := g_0, i=1, \ldots, p)$. The following Proposition 2.1. deals with a less trivial case where separate epiconvergence implies convergence in the sense of Definition 2.1.

4. Since this notion implies the epiconvergence of each scalar optimization problem $(C_n f_i^n)$ to (C, f_i), convexity is preserved, i.e., if X is a topological vector space and if each (C_n, f_i^n) is a convex optimization problem then so it is for each (C, f_i) ([2], Chap. 3) in other words if the P_n are convex MOPS so it is for P.

5. An interpretation in terms of set convergence involving the epigraphs and the complements of the strict hypographs of the $\bar{f}^n$'s and $\bar{f}$ can be given [9].

PROPOSITION 2.1. *Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space and (C, f), (C_n, f^n), $n \in N$, convex MOPS such that*

$$\forall i \in \{1, \ldots, p\}, (C_n, f_i^n) \rightarrow (C, f_i).$$

Assume further (qualification assumption) $intC \neq \emptyset$. Then

$$(C_n, f^n) \rightarrow (C, f).$$

Proof. We must only verify sentence (1) in Definition 2.1.
Let B denote the unit ball in X. Thanks to the qualification assumption and the finite dimension q of X there exist $a \in C$, $s_1 > 0$ and a qp-simplex $S = co\{y^k; k=1, \ldots, qp+1\}$ in X^p (product space endowed with the sup norm) such that

$$0 \in intS \subset S \subset int(s_1 B^p) \subset (a, \ldots, a) - C^p.$$

So there exists $z^k \in C^p$ such that

$$y^{k,i} = a - z^{k,i}, \quad k=1, \ldots, qp+1, \quad i=1, \ldots, p.$$

Then, from the convergence assumption, there exists $z_n^{k,i} \in C_n$ such that $z_n^{k,i} \rightarrow z^{k,i}$ and $limsup_{n\rightarrow\infty} f_i^n(z_n^{k,i}) \leq f_i(z^{k,i})$.

Let us set $y_n^{k,i} := a - z_n^{k,i}$ and $S_n = co\{y_n^k; k=1, \ldots, qp+1\}$. There exists $0 < s_0 < s_1$ such that for n large enough,

$$s_0 B^p \subset S_n \subset int(s_1 B^p).$$

Now let $x \in C$. For each i, there exists $x_n^i \in C_n$ such that $x_n^i \to x$ and $limsup_{n\to\infty} f_i^n(x_n^i) \leq f_i(x)$.

We shall build a sequence $\{x_n\}$ not depending on i and suitable for each i. Let $t_n = max_i \| x_n^i - x \|$. For n large enough,

$$s_0 \frac{x_n^i - x}{t_n} = a - \bar{z}_n^i ,$$

where $\bar{z}_n^i = \Sigma_{k=1}^{qp+1} \mu_{k,n} z_n^{k,i}$, for some $\mu_{k,n} \geq 0$ with $\Sigma_{k=1}^{qp+1} \mu_{k,n} = 1$. Take

$$x_n = \frac{s_0}{s_0 + t_n} x + \frac{t_n}{s_0 + t_n} a = \frac{s_0}{s_0 + t_n} x_n^i + \frac{t_n}{s_0 + t_n} \bar{z}_n^i , \quad \forall i = 1, \ldots, p .$$

We have $x_n \in C_n$ and $x_n \to x$. Moreover, because f_i^n is convex, and as $n \to \infty$, $f_i^n(x_n^{k,i})$ is bounded above and $t_n \to 0$, sentence (1) is satisfied with $\{x_n\}$. □

REMARK 2.2. In fact, Proposition 2.1 can be obtained as a corollary of a result in [3] (Theorem 1.1). The direct proof given here for the sake of selfcontaining is largely inspired from [3].

PROPOSITION 2.2. *Let $f^n, f: X \to Y$ such that f^n converges to f uniformly and f is continuous.*

Let (C_n, g^n), (C, g) be MOPS such that $(C_n, g^n) \to (C, g)$. Then

$$(C_n, f^n + g^n) \to (C, f + g) .$$

Proof. It is easy to see that under the assumptions on the f^n and f, $x = \lim_{n\to\infty} x_n$ implies $f(x) = \lim_{n\to\infty} f^n(x_n)$.

Then the conclusion follows trivially from Definition 2.1 noticing that the extension of $f^n + g^n$ (resp. $f + g$) by $+\infty$ outside C_n (resp. C) is equal to the sum of f^n (resp. f) and the extension of g^n (resp. g). □

REMARK 2.3. If X is a metric space or a topological vector space, the uniform convergence can even be assumed on the bounded subsets of X.

COROLLARY 2.1. *Assume that f^n and f are as in Proposition* 2.2. *If C is a non-empty* closed *subset of X, then*

$$(C, f^n) \to (C, f) .$$

Proof. It is enough to check that for every non-empty closed subset C of X and every objective $g: C \to Y$ such that each g_i is lower semi-continuous, we have

$$(C, g) \to (C, g) \tag{3}$$

and then apply Proposition 2.2 with $g \equiv 0$. In proving (3) the only difficulty is verification of the sentence (2) in Definition 2.1 in the case where x does not belong to C.

Let $x_n \to x$. Because C is closed, the set of indices n for which $x_n \in C$ is finite and then

$$\forall i, \liminf_{n\to\infty} \bar{g}_i(x_n) = +\infty = \bar{g}_i(x) . \qquad \square$$

REMARK 2.4. Corollary 2.1 provides a framework for approximating a MOP in which only the objective is approximated. This type of approximation, rather restrictive because it does not allow one to take into account an approximation of the feasible set (for instance by penalization of the constraints in multiobjective programming), has been considered in [5].

COROLLARY 2.2. *Let $f: X \to Y$ be a continuous mapping and $\{(C_n, g^n)\}$ a sequence of MOPS converging to some MOP (C, g). Then*

$$(C_n, f + g^n) \to (C, f + g) .$$

Proof. In Proposition 2.2, take $f^n = f$. $\square$

Concerning the relationship between the convergence of a sequence $\{P_n := (C_n, f^n)\}$ of MOPS to the MOP $P := (C, f)$ and the epiconvergence of the scalarized problems $P_{n,y} := (C_n, y.f^n(.))$ where y is given in Y, we have the following easily proven result.

PROPOSITION 2.3. *If $P_n \to P$ then for all y in Y_+, $P_{n,y}$ epiconverges to P_y.*

We shall state now the three main results dealing with approximation of the efficient set of a MOP in the weak or strong Pareto sense.

THEOREM 2.1. *Let $\{P_n := (C_n, f^n)\}$ be a sequence of MOPS such that $P_n \to P := (C, f)$. Consider a sequence $\{\epsilon^n\} \subset Y$ with $\lim_{n\to\infty} \epsilon^n = 0$, and sequences $\{n_k\} \subset N$ and $\{\bar{x}_k\} \subset X$ such that $\bar{x}_k \in \epsilon^{n_k} - E_w(P_{n_k})$, $\bar{x}_k$ converging to some $\bar{x}$. Then $\bar{x} \in E_w(P)$.*

Proof. Let $x \in C$. Consider a sequence $\{x_n\}$ associated with x by Definition 2.1, sentence (1). We have

$$\forall k \in N , \quad \exists i_k , \quad f_{i_k}^{n_k}(\bar{x}_k) \leq f_{i_k}^{n_k}(x_{n_k}) + \epsilon_{i_k}^{n_k} .$$

In fact, as the set of indices i is finite, we can assume (after another extraction of a subsequence) that i_k does not depend on k. Let us call it i. Then

$$\forall k \in N , \quad f_i^{n_k}(\bar{x}_k) \leq f_i^{n_k}(x_{n_k}) + \epsilon_i^{n_k} . \tag{4}$$

Now define the sequence $\{\tilde{x}_n\}$ by

$$\tilde{x}_n = \begin{cases} \bar{x}_k & \text{if } n = n_k , \\ \bar{x} & \text{otherwise} . \end{cases}$$

Clearly $\tilde{x}_n \to \bar{x}$. From (4) and since $\lim_{n\to\infty} \epsilon^n = 0$,

$$\bar{f}_i(\bar{x}) \leq \liminf_{n\to\infty} \bar{f}_i^n(\tilde{x}_n) \leq \cdots$$
$$\liminf_{k\to\infty} f_i^{n_k}(\bar{x}_k) \leq \limsup_{k\to\infty} f_i^{n_k}(x_{n_k}) \leq \limsup_{n\to\infty} f_i^n(x_n) \leq f_i(x) . \quad (5)$$

In (5) the extreme inequalities are obtained from Definition 2.1 (sentence (1) yields the right one, sentence (2) yields the left one). As $f_i(x)$ is finite (5) implies that $\bar{f}_i(\bar{x})$ is finite, i.e., $\bar{x} \in C$ and $\bar{f}_i(\bar{x}) = f_i(\bar{x})$. So

$$\forall x \in C , \quad \exists i , \quad f_i(\bar{x}) \leq f_i(x), \quad \text{i.e., } \bar{x} \in E_w(P) . \qquad \square$$

REMARK 2.5. The analogous result for *strong* Pareto efficiency is not true in general. Otherwise, for every MOP $P := (C, f)$ where C is compact and f continuous, as $E_s(P) \neq \emptyset$ and (see the proof of Corollary 2.1) $P \to P$, $E_s(P)$ would be (sequentially) closed, which is not true in general. However, in some special cases, Theorem 2.1 for *strong* Pareto efficiency holds true, for instance if X is a vector space, C is convex and all the f_i are *strictly* convex. In this case $E_s(P) = E_w(P)$ and the result comes from Theorem 2.1 and Proposition 1.1 (3). This has already been noticed in ([11] Theorem 2.2), in the special context of penalization and for exact efficient points, with a direct proof. Another special case is given by the following proposition.

PROPOSITION 2.4. *Same assumptions as in Theorem* 2.1 *except* $\{\bar{x}_k\}$ *being defined by* (*let* $y > 0$)

$$\bar{x}_k \in y.\epsilon^{n_k} - Argmin\ P_{n,y}$$

(*therefore, by proposition* 1.2(1), $\bar{x}_k \in \epsilon^{n_k} - E_s(P_k)$). *Then* $\bar{x} \in E_s(P)$.

Proof. By Proposition 2.3, $P_{n,y}$ epiconverges to P_y. Then, by Theorem 2.1 itself applied to the scalarized problems taking account of Remark 1.1 (4), we get $\bar{x} \in Argmin\ P_y$. Then by Proposition 1.2(1), $\bar{x} \in E_s(P)$. $\square$

The second main result is, in some sense, the converse of the first one, stating that every weak (therefore strong) Pareto efficient point of P can be obtained as a limit of a sequence of ϵ^n-strong Pareto efficient points of P_n.

THEOREM 2.2 *Let* $\{P_n := (C_n, f^n)\}$ *be a sequence of MOPS such that* $P_n \to P := (C, f)$. *Assume there exists a relatively sequentially compact subset D of X such that, for all n,* $C_n \subset D$. *Then, if* $E_w(P) \neq \emptyset$, *for any* $\bar{x} \in E_w(P)$ *there exists a sequence of* $\epsilon^n > 0$ *with* $\lim_{n\to\infty} \epsilon^n = 0$ *and a sequence* $\{\bar{x}_n\}$ *such that* $\lim_{n\to\infty} \bar{x}_n = \bar{x}$ *and* $\bar{x}_n \in \epsilon^n - E_s(P_n)$ *for all n large enough.*

Proof. Consider a sequence $\{\bar{x}_n\}$ associated with nx by Definition 2.1, sentence (1). We follow the method of proof given in ([2] Theorem 2.2 (ii)) for the scalar case. We shall first prove the following property

$$\forall \epsilon > 0 , \quad \exists n_\epsilon , \quad \forall n \geq n_\epsilon , \quad \bar{x}_n \in \epsilon - E_s(P_n) , \quad (6)$$

otherwise

$$\exists \epsilon^0 > 0 , \quad \exists \{n_k\} , \quad \forall k , \quad \bar{x}_{n_k} \not\in \epsilon^0 - E_s(P_{n_k}) ,$$

hence

$$\exists \epsilon^0 > 0 , \quad \exists \{n_k\} , \quad \forall k , \quad \exists x_k \in C_{n_k} , \quad \forall i , \quad f_i^{n_k}(\bar{x}_{n_k}) \geq f_i^{n_k}(x_k) + \epsilon_i^0 . \tag{7}$$

The sequence $\{x_k\}$ contained in D is relatively sequentially compact. So there exists a subsequence $\{n_h\}$ of $\{n_k\}$ and $x \in X$ such that $x_h \to x$. From (7) and Definition 2.1 we get

$$\forall i , \quad f_i^{n_h}(\bar{x}_{n_h}) \geq f_i^{n_h}(x_h) + \epsilon_i^0 = \bar{f}_i^{n_h}(x_h) + \epsilon_i^0$$

and

$$f_i(\bar{x}) \geq \limsup_{h\to\infty} f_i^{n_h}(\bar{x}_h) \geq \liminf_{h\to\infty} \bar{f}_i^{n_h}(x_h) + \epsilon_i^0 \geq \bar{f}_i(x) + \epsilon_i^0 .$$

Therefore, $\bar{f}_i(x)$ is finite, i.e., $x \in C$. Furthermore

$$\forall h , \quad \exists i_h , \quad f_{i_h}^{n_h}(x_h) \leq f_{i_h}^{n_h}(\bar{x}_{n_h}) + f_{i_h}(x) - f_{i_h}(\bar{x}) - \epsilon_{i_h}^0 , \tag{8}$$

otherwise, by (7) we would have

$$\forall i , \quad f_i(\bar{x}) > f_i(x) , \quad \text{i.e., } \bar{x} \not\in E_w(P) .$$

In (8) we can assume, since the set of indices i is finite, (after another extraction of a subsequence) that i_h does not depend on h. Let us call it i. Then

$$\forall h , \quad f_i^{n_h}(x_h) \leq f_i^{n_h}(\bar{x}_h) + f_i(x) - f_i(\bar{x}) - \epsilon_i^0 . \tag{9}$$

From (9) and Definition 2.1 we get finally

$$\exists i , \quad \epsilon_i^0 \leq 0 \text{ contrary with } \epsilon^0 > 0 .$$

So (6) is proved.

Consider now a sequence of $\epsilon^\nu > 0$ such that $\lim_{\nu\to\infty} \epsilon^\nu = 0$. From (6) we get

$$\forall \nu , \quad \exists N_\nu , \quad \forall n \geq N_\nu , \quad \bar{x}_n \in \epsilon^\nu - E_s(P_n) ,$$

where the map $\nu \to N_\nu$ can be assumed strictly increasing.

For every $n \geq N_1$ there exists a unique ν such that $N_\nu \leq n < N_{\nu+1}$. Calling it $\nu(n)$ and setting $\epsilon^n = \epsilon^{\nu(n)}$ we have

$$\forall n \geq N_1 , \quad \bar{x}_n \in \epsilon^n - E_s(P_n)$$

and, because the map $n \to \nu(n)$ is increasing, $\lim_{n\to\infty} \epsilon^n = 0$. □

For the strong Pareto case, the following third main result extends a one of ([11] Theorem 2.1) stated in the specific context of penalization (see Section 3 below, Proposition 3.4).

THEOREM 2.3. *Let $\{P_n := (C_n, f^n)\}$ be a sequence of MOPS such that $P_n \to P := (C, f)$. Assume there exists a relatively sequentially compact subset D of X such that, for all n, C_n is closed and contained in D, and f^n is lower semi-continuous (i.e., each component f_i^n is a real lower semi-continuous function). Then, $E_s(P) \neq \emptyset$, and for any $\bar{x} \in E_s(P)$ there exists a sequence $\{\bar{x}_n\} \subset X$ such that*

1. *$\forall n$, $\bar{x}_n \in E_s(P_n)$ and $\lim_{n\to\infty} f^n(\bar{x}_n) = f(\bar{x})$.*
2. *The set of limit points of subsequences of $\{\bar{x}_n\}$ is nonempty, contained in C and its image by f is $\{f(\bar{x})\}$ (hence it is contained in $E_s(P)$).*

Proof. Definition 2.1 (1) implies $C \subset D$. Then $E_s(P) \neq \emptyset$.
Let $\bar{x} \in E_s(P)$ and $\{x_n\}$ associated with $\bar{x}$ in Definition 2.1, sentence (1). For all n, there exists $\bar{x}_n \in E_s(P_n)$ such that

$$f^n(\bar{x}_n) \leqq f^n(x_n)\,. \tag{9'}$$

Since $\{\bar{x}_n\} \subset D$, there exists a convergent subsequence.

Consider such a subsequence $\{\bar{x}_{n_k}\}$ and x^0 its limit point. Define the sequence $\{\tilde{x}_n\}$ by

$$\tilde{x}_n = \begin{cases} \bar{x}_{n_k} & \text{if } n = n_k\,, \\ x^0 & \text{otherwise}\,. \end{cases}$$

By Definition 2.1 and (9'), by lines of reasoning similar to those used previously to derive (5), we get

$$\forall i,\ \bar{f}_i(x^0) \leqslant \liminf_{n\to\infty} \bar{f}_i^n(\tilde{x}_n) \leqslant \liminf_{k\to\infty} f_i^{n_k}(\bar{x}_{nk}) \leqslant \limsup_{k\to\infty} f_i^{n_k}(\bar{x}_{n_k}) \leqslant \cdots$$
$$\cdots \leqslant \limsup_{k\to\infty} f_i^{n_k}(x_{n_k}) \leqslant \limsup_{n\to\infty} f_i^n(x_n) \leqslant f_i(\bar{x})\,.$$

Therefore, $\bar{f}_i(x^0)$ is finite, i.e., $x^0 \in C$, and $f(x^0) \leqq f(\bar{x})$. Hence $f(x^0) = f(\bar{x})$ and $\lim_{k\to\infty} f^{n_k}(\bar{x}_{n_k}) = f(\bar{x})$.

We have just proved that from any subsequence of $\{f^n(\bar{x}_n)\}$, we can extract a subsequence which converges to $f(\bar{x})$. Therefore, $\lim_{n\to\infty} f^n(\bar{x}_n) = f(\bar{x})$. □

3. Application to Penalty Methods

Let $P := (C, f)$ be a MOP where f is continuous. Assume there exists a positive continuous function: $\phi : X \to R$ and a closed subset $D \subset X$ such that

$$C = \{x \in D;\ \phi(x) = 0\}\,.$$

Notice that C is then closed.

Classical exterior penalty functions used in mathematical programming enter into this framework [13].

For $r > 0$ we consider the MOP $P_r := (D, f^r)$ where

$$f_i^r := f_i + r\phi\,, \quad i = 1, \ldots, p$$

PROPOSITION 3.1. *Let $\{r_n\}$ be a sequence of positive reals such that* $\lim_{n\to\infty} r_n = +\infty$. *Let $P_n := P_{r_n}$. Then*

$$P_n \to P .$$

Proof. By Corollary 2.2 and Remark 2.1 (3), it suffices to prove the convergence of the sequence of scalar optimization problems $(D, r_n\phi)$ to the scalar optimization problem (C, g) where $g \equiv 0$. This is well-known in epiconvergence theory [2]. Nevertheless, for the sake of selfcontaining, let us give here a simple proof. For this, sentence (1) of Definition 2.1 is trivially satisfied with $x_n = x$. Sentence (2) is also trivially satisfied in the case where x belongs to C because $\bar{g}(x) = 0$ and $\overline{r_n\phi}(z) \geq 0$ for every $z \in X$. If x does not belong to C, then $\bar{g}(x) = +\infty$ and, as C is closed, for n large enough x_n does not belong to C. Notice that

$$(z \not\in C \text{ and } z \in D) \Rightarrow \phi(z) > 0 \tag{10}$$

If the set of indices for which $x_n \in D$ is finite, then, for n large enough $\overline{r_n\phi}(x_n) = +\infty$ and sentence (2) is satisfied. If this set is not finite there exists a subsequence $\{n_k\}$ such that $x_{n_k} \in D$. Therefore $x \in D$ because D is closed and $\overline{r_{n_k}\phi}(x_{n_k}) = r_{n_k}\phi(n_{n_k})$ which converges to $+\infty$ because ϕ is continuous and $\phi(x) > 0$. Finally

$$\liminf_{n\to\infty} \overline{r_n\phi}(x_n) = +\infty . \qquad \square$$

The following three results come directly from Proposition 1.3 (2), Proposition 3.1 and, respectively, Theorems 2.1, 2.2, and 2.3. The details of proof are left to the reader. We assume in addition that D is sequentially compact.

PROPOSITION 3.2. *Consider a sequence $\{\epsilon^n\} \subset Y/\{0\}$ such that $\epsilon^n \not< 0$ and* $\lim_{n\to\infty} \epsilon^n = 0$. *Then*

1. $\forall n$, $\epsilon^n - E_w(P_n) \neq \emptyset$.
2. *for any $\{\bar{x}_n\}$ such that $\forall n\, \bar{x}_n \in \epsilon^n - E_w(P_n)$, the set of limit points of subsequences of $\{\bar{x}_n\}$ is nonempty and contained in $E_w(P)$.*

PROPOSITION 3.3. $E_w(P) \neq \emptyset$ *and* $\forall \bar{x} \in E_w(P)$, $\exists\{\epsilon^n\} \subset Y$, $\epsilon^n > 0$, $\lim_{n\to\infty} \epsilon^n = 0$ *such that $\bar{x} \in \epsilon^n - E_w(P_n)$ for n large enough.*

PROPOSITION 3.4. $E_s(P) \neq \emptyset$ *and* $\forall \bar{x} \in E_s(P)$, $\exists\{\bar{x}_n\} \subset D$ *such that*

1. $\forall n$, $\bar{x}_n \in E_s(P_n)$ *and* $\lim_{n\to\infty} f^n(\bar{x}_n) = f(\bar{x})$.
2. *The set of limit points of subsequences of $\{\bar{x}_n\}$ is nonempty, contained in $E_s(P)$ and its image by f is $\{f(\bar{x}).\}$*

REMARK 3.1. Proposition 3.2 above has already been obtained in [11] with $\epsilon^n = 0$ and under additional convexity assumption.

The two following *finite* convergence results, specific to the penalty method and useful in *integer* multiobjective programming, slightly extend a one of ([11] Theorem 3.1).

PROPOSITION 3.5. *Assume D is finite.*

Let $\sigma := min_{x \in D/C} \phi(x)$ *and* $M := max_{x,x' \in D} max_i(f_i(x) - f_i(x'))$. *Then*

$$\forall \epsilon \geqq 0\,, \quad \forall r > (M + max_i \epsilon_i)/\sigma\,, \quad \epsilon - E_w(P_r) = \epsilon - E_w(P)$$
$$\epsilon - E_s(P_r) = \epsilon - E_s(P)\,.$$

Proof. 1. $\epsilon - E_w(P_r)$ (*resp.* $\epsilon - E_s(P_r)) \subset \epsilon - E_w(P)$ (*resp.* $\epsilon - E_s(P)$).

Let $x_r \in \epsilon - E_w(P_r)$. Then $x_r \in D$ and we have

$$\forall x \in C\,, \quad \exists i_x \text{ such that } f_{i_x}(x_r) + r\phi(x_r) \leqslant f_{i_x}(x) + \epsilon_{i_x}\,.$$

Then

$$r\phi(x_r) \leqslant M + max_i\, \epsilon_i\,.$$

Therefore, $\phi(x_r) < \sigma$. Then $x_r \in C$ and hence $x_r \in \epsilon - E_w(P)$ (resp. $\epsilon - E_s(P)$ if $x_r \in \epsilon - E_s(P_r)$).

2. $\epsilon - E_w(P) \subset \epsilon - E_w(P_r)$; $\epsilon - E_s(P) \subset \epsilon - E_s(P_r)$.

Let $\bar{x} \in \epsilon - E_w(P)$ (resp. $\epsilon - E_s(P)$). We have $\bar{x} \in C$ and

$$\forall x \in C\,, \quad f(\bar{x}) \not> (resp.\ \not\geq) f(x) + \epsilon. \tag{11}$$

Now let $x \in D/C$. By definition of M and σ, we have

$$\forall i\,, \quad f_i(x) + r\phi(x) + \epsilon_i \geqslant f_i(x) + r\sigma + \epsilon_i > f_i(\bar{x}) + 2\epsilon_i \geqslant f_i(\bar{x})$$

which joined to (11) gives

$$\forall x \in D\,, \quad f^r(\bar{x}) \not> (\text{resp. } \not\geq) f^r(x) + \epsilon\,,$$

i.e., $\bar{x} \in \epsilon - E_w(P_r)$ (resp. $\epsilon - E_s(P_r)$). □

COROLLARY 3.1. *Under assumptions of Proposition* 3.5, *let sequences* $\{\epsilon^n\} \subset Y_+$, $\{r_n\} \subset R_+$, *such that* $lim_{n\to\infty} \epsilon^n = 0$ *and* $lim_{n\to\infty} r_n = +\infty$. *Then, for n large enough*

$$\epsilon^n - E_w(P_n) = E_w(P)\,.$$

Proof. Let $\epsilon > 0$ defined in Proposition 1.4. We have

$$\exists N_1 \text{ such that } \forall n \geqslant N_1\,, \quad \epsilon^n \leqq \epsilon$$
$$\exists N_2 \text{ such that } \forall n \geqslant N_2\,, \quad r_n > (M + max_i \epsilon_i)/\sigma\,.$$

Then, $\forall n \geqslant max\{N_1, N_2\}$ $r_n > (M + max_i \epsilon_i^n)/\sigma$.

Hence $\epsilon^n - E_w(P_n) = \epsilon^n - E_w(P) = E_w(P)$, the last equality coming from Proposition 1.4. □

4. Application to Variational Approximation

Along this section X is assumed to be a real reflexive Banach space, C_n, $n \in N$ and C are non-empty closed convex subsets of X and $f: X \to Y$ is convex and norm continuous.

DEFINITION 4.1. (see [2]). We say that C_n converges to C in the Mosco sense and we denote $C_n \xrightarrow{M} C$ if the two following sentences hold.

$$\forall x \in C, \quad \exists \{x_n\} \subset X \text{ such that } \forall n, \quad x_n \in C_n, \text{ and } x = s - \lim_{n \to +\infty} x_n, \tag{12}$$

$$\begin{aligned} &\forall x \in X, \quad \forall \{n_k\}, \quad \forall \{x_k\} \subset X \text{ such that } \forall k, \quad x_k \in C_{n_k} \\ &\quad \text{and } x = w - \lim_{k \to +\infty} x_k, \text{ then } x \in C, \end{aligned} \tag{13}$$

where s (resp. w) refers to the norm (resp. weak) topology of X.

PROPOSITION 4.1. *Let $P_n := (C_n, f)$ and $P := (C, f)$.*
If $C_n \xrightarrow{M} C$ then $P_n \to P$ for both norm and weak topologies of X.

Proof. f being norm continuous, (12) implies sentence (1) of Definition 2.1 for the norm (hence for the weak) topology. Now let us rewrite in the present situation, sentence (2) of Definition (2.1) under the following equivalent setting

$$\forall x \in X, \ \forall \{n_k\}, \ \forall \{x_k\} \subset X \text{ such that } \forall k \ x_k \in C_{n_k} \text{ and } x = \lim_{k \to \infty} x_k,$$

then

$$\forall i, \quad \liminf_{k \to \infty} f_i(x_k) \begin{cases} \geqslant f(x) & \text{if } x \in C, \\ = +\infty & \text{if } x \notin C. \end{cases} \tag{14}$$

Then, since each f_i is lower semicontinuous for the weak topology, (13) implies (14) for the weak (hence for the norm) topology. □

REMARK 4.1. When $p = 1$, this type of approximation of (C, f) by (C_n, f) reduces to the abstract formulation of discretization methods like finite elements method for convex variational problems ([4], Chap. 4).

Hereafter we assume that C is bounded, that for all n, $C_n \subset D$ where D is a bounded subset of X and that $C_n \xrightarrow{M} C$. Then, noticing that in a reflexive Banach space a bounded subset is weakly relatively sequentially compact, that a closed convex subset is weakly closed and that a norm continuous real convex function is weakly lower semi-continuous, the three following results come directly from Proposition 1.3 (2), Proposition 4.1 and, respectively, Theorems 2.1, 2.2, 2.3, X being equipped with the *weak* topology.

PROPOSITION 4.2. *Same statement as Proposition* 3.2.

PROPOSITION 4.3. *$E_w(P) \neq \emptyset$ and $\forall \bar{x} \in E_w(P)$, $\exists \{\epsilon^n\} \subset Y$, $\epsilon^n > 0$, with* $\lim_{n\to\infty} \epsilon^n = 0$ *and $\exists \{\bar{x}_n\} \subset X$ such that* $\lim_{n\to\infty} \bar{x}_n = \bar{x}$ *and $\bar{x}_n \in \epsilon^n - E_w(P_n)$ for n large enough.*

PROPOSITION 4.4. *Same statement as Proposition* 3.4.

5. Conclusion

This paper deals with results concerning the approximation of both efficient solutions and multiobjective optimization problems with applications to exterior penalization and to variational approximation. The main tools introduced to do that are, on the one hand, a notion of approximate efficient solution which reduces to the one of suboptimal solution in scalar optimization and, on the other hand, a kind of equi-epiconvergence of a finite number of real functions. Most variational properties of epiconvergence are thus extended to multiobjective optimization.

As in the scalar case such type of approximation aims to replace an optimization problem by simpler ones for the computational point of view. For instance, in multiobjective programming the penalty results of Section 3 can be used to remove nonlinear constraints.

Thus, although the present work is rather of theoretical interest, its results may be of potentially practical benefit in designing effective numerical methods for computing compromise solutions to optimization models of multicriteria decision making.

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References

1. Aubin, J. P. (1979), *Mathematical Methods of Game and Economic Theory*, North Holland Publ. Amsterdam.
2. Attouch, H. (1984), *Variational Convergence for Functions and Operators*, Pitman Advanced Publishing Program, Boston-London-Melbourne.
3. Aze, D. and J. P. Penot (1987), Operations on Convergent Families of Sets and Functions, Publication AVAMAC, Université de Perpignan, France, no. 87–05, Vol. 1.
4. Céa, J. (1971), *Optimisation, théorie et algorithmes*, Dunod.
5. Dugošija, D. P. and M. D. Ašić (1986), Uniform Convergence and Pareto Optimality, *Optimization* 17 (6), 723–729.
6. Karlin, S. (1960), *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. I, McGraw Hill, New York.
7. Kutateladze, S. S. (1979), Convex ϵ-Programming, *Soviet Mathematical Doklady* **20**, 391–393.

8. Lemaire, B. (1988), Coupling Optimization Methods and Variational Convergence, *International Series of Numerical Mathematics*, Vol. 84(c), Birkhäuser Verlag, pp. 163–179.
9. Lemaire, B. (1987), Convergence in Vector Optimization. Communication at the *Congrès Franco-Québécois d'Analyse Non-Linéaire*, Perpignan, France.
10. Vály, István (1987), Epsilon Solutions and Duality in Vector Optimization, Working paper WP-87-43. Institute for Applied Systems Analysis. Laxenbourg, Austria.
11. White, D. J. (1984), Multiobjective Programming and Penalty Functions, *JOTA* **43** (4), 583–599.
12. White, D. J. (1986), Epsilon Efficiency, *Journal of Optimization Theory and Application* **49** (2), 319–337.
13. Zangwill, W. I. (1969), *Nonlinear Programming*, Prentice-Hall, Englewood Cliffs, New Jersey.